what is Topology ?

Basically, topology is the modren version of geometry.
The idea is that if one geometric object can be continuously transformed into another then the two objects are to be viewed as being top the sam

- In ordinary Euclidean geometry, you can move things around and flip them over, but you can't streach or bend them.

In topology, any continuous change is allowed. So a circle is the same as a triangle or a squar, because you just pull on a parts of the circle to make a corners and then straighten the sides, to change a circle into a squar.

- The circle isnt the same as a figer 8, because although you can squash the middle of a circle together to make it into a figer 8 continuously, when you try to undo it, you have to breack the connection in the middle and this is discontinuous. (Ex/ a plate and powl are the same topologically)

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General topology (point set topology) may be defined as

a set of points, along with a set of neighbourhoods for each points, satisfying a set of axioms relating points and neighbourhoods.

Preliminaries From Set Theory

Notations; If A and B are sets then

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$, more generally,

If $A_\alpha, \alpha \in \Lambda$ is a family of sets, then

$\bigcup_\alpha A_\alpha = \{x ; x \in A_\alpha \text{ for some } \alpha\}$

$\bigcap_\alpha A_\alpha = \{x ; x \in A_\alpha \text{ for every } \alpha\}$

$A \times B = \{(a,b) ; a \in A , b \in B\}$.

If $A \subseteq B$, then $B - A = \{x ; x \in B \text{ and } x \notin A\}$.

what the difference between Topology and metric space?

Ans: Both metric spaces and topology are useful for defining continuity. Every metric space can be made a top. so that the notion of cont in metric spaces agrees with the notion of cont in top. The reverse is not true. The notion of topology is more general.

Def: If X is any set, any subset of $X \times X$ is called a relation on X.

Def: A relation R on X is called an equivalence relation if R satisfies the following axioms:

(i) R is reflexive, i.e $\forall a \in X$, $(a,a) \in R$

(ii) R is symmetric, i.e if $(a,b) \in R$, then $(b,a) \in R$

(iii) R is transitive, i.e if $(a,b) \in R$ and $(b,c) \in R$ $\Rightarrow (a,c) \in R$

Def:- Two sets A and B are said to be equivalent or have the same cardinal number if $\exists$ a 1-1 onto mapping f from A onto B.

Def:- A set A is called infinite if A is equivalent to a proper subset of itself. A set A is called finite if A is not infinite. For example, the set $N = \{1, 2, 3, \ldots\}$ of natural number is infinite since the mapping f defined by $f(n) = 2n$ is a 1-1 onto map from N onto its subset of even natural number.

Def/ (1) Any set X with cardinality less than that $\mathbb{N}$

i.e $|X| < |\mathbb{N}|$ is a finite set

(2) 〃 〃 〃 〃 〃 $|X| = |\mathbb{N}|$ is countably infinite set

(3) 〃 〃 〃 〃 〃 $|X| > |\mathbb{N}|$ is uncountable.

Ex/ $|\mathbb{R}| = c > |\mathbb{N}|$ is uncountable.

Def: Let X be any non empty set. A topology on X is a collection T of subsets of X, such that:

1) ϕ and $X \in T$

2) The union of any collection of elements of T is an element of T (i.e if $U_i \in T$ then $\bigcup_{i \in T} U \in T$)
   $i \in T$

3) The intersection of any finite collection of elements of T is an element of T
(i.e if $U_1, U_2, \dots, U_n \in T$ then $U_1 \cap U_2 \cap \dots \cap U_n \in T$)

Def: The elements of T are called open sets

Def: A top space (X, T) is a set X together with a topology T on X.

Ex) Let $X = \{a, b, c\}$

$T = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ $\therefore$ T is top on X

$T' = \{\phi, X, \{a\}, \{b\}, \{b, c\}\}$ is not top on X
since $\{a\} \cup \{b\} = \{a, b\} \notin T$

$T'' = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

called discrete top on X. (largest top)
$2^X = 2^3 = 8$

$T^{(1)} = \{\Phi, X\}$ indiscrete top or (trival top) or (smallest top).

Note: Any set has more than one elt. has at least two top discrete and indiscrete top.

EX1 Let $X = \{1, 2\}$. There are four possible top on X

(1) $T = \{\Phi, X\}$ ⟵ trival one

(2) $T = \{\Phi, X, \{1\}, \{2\}\}$ dis. top

(3) $T = \{\Phi, X, \{1\}\}$

(4) $T = \{\Phi, X, \{2\}\}$.

HW(1): Let $X = \{1, 2, 3\}$. There are 29 different top. on X

(1) $T_1 = \{\Phi, X\}$

(2) $T_2 = P(X)$ (with 8 elt)

(3) $T_3 = \{\Phi, X, \{1\}\}$

(4) $T_4 = \{X, \Phi, \{1, 2\}\}$

$T_5 = \{\phi, X, \{1\}, \{1, 2\}\}$

$T_6 = \{\phi, X, \{3\}, \{1, 2\}\}$

$T_7 = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$

$T_8 = \{\phi, X, \{1\}, \{1, 2\}, \{1, 3\}\}$.

HW2

Ex1 Let $X = \mathbb{N}$, Let $A_n = \{1, 2, \dots, n\}$ $n = 1, 2, 3, \dots$

i.e $A_1 = \{1\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2, 3\}, \dots$

$T = \{\phi, X, A_1, A_2, \dots, A_n, A_{n+1}, \dots\}$

(1) $\phi, X \in T$

(2) Let $A_i, A_j \in T$ to prove $A_i \cap A_j \in T$

$$A_i \cap A_j = \begin{cases} A_i & \text{if } i < j \\ A_j & \text{if } j < i \end{cases} \in T$$

3) Let $A_i, A_j \in T$.

$$A_i \cup A_j = \begin{cases} A_i & \text{if } i > j \\ A_j & \text{if } j > i \end{cases}$$

$\therefore \bigcup_{i \in I} A_i \in T$

Hence T is topology on X

Def: In a topological space, any open subset A is a neighborhood for each elements.

Def: Let $X=\mathbb{R}$. then U be an open subset of $\mathbb{R}$ iff $\exists$ $\epsilon>0$ s.t $(x-\epsilon, x+\epsilon)\subseteq U$ $\forall x\in U$.

EX: Let $X=\mathbb{R}$, $T=\{\phi, U \text{ all open subset of } \mathbb{R}\}$ then prove that T is a topological on $\mathbb{R}$.

1) $\phi, \mathbb{R}\in T$

2) let $U_1, U_2\in T$, to prove $\bigcap_{i=1}^{n} U_i\in T$

let $x\in\bigcap U_i \Rightarrow x\in U_i\ \forall i$

$\xRightarrow{\text{open } U_j\text{'s}} \exists\ \epsilon>0$ s.t $(x-\epsilon, x+\epsilon)\subset U_i\ \forall i$

$\Rightarrow (x-\epsilon, x+\epsilon)\subseteq\bigcap_{i=1}^{n} U_i$

$\therefore \bigcap_{i=1}^{n} U_i$ is open and an elt. of T

3) Let $\{U_i\}_{i\in I}\in T$, to prove $\bigcup U_i\in T$

let $x\in\bigcup_{i\in I} U_i \Rightarrow x\in U_i$ for some i

$\Rightarrow \exists\ \epsilon>0$ s.t $(x-\epsilon, x+\epsilon)\subset U_i$

$\therefore (x-\epsilon, x+\epsilon)\subset\bigcup_{i\in I} U_i$

$\therefore \bigcup U_i\in T$

$\therefore T$ is top on $\mathbb{R}$ and called the usual top.

H.W Let X be any infinite set.

$T = \{ U \mid U \subseteq X, X - U \text{ is finite} \} \cup \{ \phi \}$

Show that T is topology on X.

Sol: $\phi, X \in T$

let $A, B, C \dots \in T$

$\Rightarrow A \cup B \cup C \dots \subseteq X$ because

$X - (A \cup B \cup C \dots) = (X-A) \cap (X-B) \cap (X-C) \dots$

since each of $(X-A), (X-B), (X-C), \dots$ is finite

$\therefore$ The intersection is finite

$\therefore A \cup B \cup C \dots \in T$

Let $A_1, A_2, \dots, A_n \in T$

look at $A_1 \cap A_2 \dots \cap A_n$

if any A_n is empty, $A_1 \cap A_2 \dots A_n$ is empty

Assume none of the A_n's is empty.

$X - (A_1 \cap A_2 \cap A_3 \dots \cap A_n) = (X - A_1) \cup (X - A_2) \dots \cup (X - A_n)$

since each of $(X-A_1), \dots, (X-A_n)$ is finite

$\therefore (X-A_1) \cup \dots \cup (X-A_n)$ is finite, because it is a finite union of finite sets.

$\therefore (X, T)$ is a topological space.

The topology T is called the cofinite topology on X

Def: Let (X,T) be a topological space. a subset E of X is called a closed set iff $X-E$ is open (i.e $X-E \in T$).

H.W

Proposition: Let (X,T) be a topological space then the following are satisfied:

1) ϕ, X are closed
2) The finite union of any collection of closed set is closed
3) The intersection of any collection of closed set is closed.

proof: 1) $\phi = X - X$, $X = X - \phi$

$\therefore$ ϕ and X are closed

2) Let $A_1, A_2, \ldots, A_n$ be closed

$\Rightarrow A_i = X - E_i$ where E_i is open

$A_1 \cup A_2 \ldots \cup A_n = (X-E_1) \cup (X-E_2) \ldots$

$\cup (X-E_n) = X - (E_1 \cap E_2 \ldots \cap E_n)$

where $(E_1 \cap E_2 \cap \cdots \cap E_n)$ is open because it is a finite intersection of open sets.

(3) Let $\{E_\alpha\}_{\alpha \in \Lambda}$ be a collection of closed sets

consider : $X - (\bigcap_\alpha E_\alpha) = \bigcup_\alpha (X - E_\alpha)$ and this open ?

This define a topology

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EXC: Show by an example that the intersection of an infinite collection of open sets may not be open, and the union of an infinite number of closed sets may not be closed?

Hint :- Let $X = \mathbb{N}$, T cofinite topology on X

Sol: Let $X = \mathbb{N}$ and T be cofinite topology on X

for each number n, define the set S_n as follows:

$S_n = \{1\} \cup \{n+1\} \cup \{n+2\} \cup \{n+3\} \cup \dots$

clearly, each S_n is an open set in the topology T since its complement is finite set

$$\bigcap_{n=1}^{\infty} S_n = \{1\}$$

As the complement of $\{1\}$ is neither $\mathbb{N}$ or a finite set so $\{1\}$ is not open

Similarly for the other part.

Def: Let (X,T) be a top. space, $p \in X$, any open set containing p is called a neighbourhood of p

Def: Let $E \subseteq X$ (top), $p \in E$, p is called an interior point of E iff $\exists$ a neighbourhood N_p of p in X s.t $N_p \subseteq E$. The set of all interior points of E is called the interior of E and is denoted by $i(E)$.

Remark:- A subset $E \subseteq X$ is open iff $i(E) = E$ (which is the largest open set)
i.e iff every point of E is an interior point

Proof: Let $p \in E$, Assume E is open
take E as a neighbourhood.

$\Leftarrow$ **Exc:** Suppose $i(E) = E$

$\forall\ p \in E = i(E) \Rightarrow \exists$ by def above a neigh N_p of p which is an open set s.t $p \in N_p \subseteq E$

so $E = \bigcup_{p \in E} N_p$, i.e E is an open

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Def; Let $E \subseteq X$, and $p \in X$, p is called a limit point of E iff every neigh N_p of p contains at least one point of E different from p i.e $(N_p - \{p\}) \cap E \neq \phi$

The set of all limit points of E is called the derived set of E and is denoted by $d(E)$.

$E \cup d(E) = \overline{E}$ = closure of E

Def: A point $p \in X$ is called a boundary point of E iff every neighbourhood of p has non-empty intersection with both E and $X - E$.

The set of all boundary points of E is called the boundary of E, it is denoted by $b(E)$.

Ex1 Let $X = \{a, b, c, d\}$

$T = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$

$S = \{a, c\}$

To find $i(S)$; a is an interior of S because $a \in \{a\} \subseteq S$ where $\{a\}$ is a neigh of a.

c is not an interior point of S

$\therefore i(S) = \{a\}$.

To find $d(S)$: a is not a limit point of S because $(\{a\} - \{a\}) \cap S = \phi$

b is not a limit point of S because $(\{b\} - \{b\}) \cap S = \phi$

$c \in d(S)$, $d \in d(S)$

$\therefore d(S) = \{c, d\}$.

$\bar{S} = S \cup d(S) = \{a, c, d\}$

$b(S)$; $a \notin b(S)$, $b \notin b(S)$, $c \in b(S)$, $d \in b(S)$

$\therefore b(S) = \{c, d\}$.

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H.W $X = \mathbb{N}$ = natural numbers
T = The cofinite top.

$E = (2, 4, 6, \dots)$, even natural numbers.

Find $i(E)$ and $d(E)$

Sol: $2 \notin i(E)$ because any subset contained in E is not an open subset; its complement is infinite.

$\therefore i(E) = \phi$

To prove $2 \in d(E)$, assume $2 \notin d(E)$

$\Rightarrow \exists N(2)$ s.t $N(2)$ does not contain any even natural number, which implies $X - N(2)$ is infinite,

Hence $X - N(2)$ is not open set $\times$ contradiction

$\therefore 2 \in d(E)$

Similarly $n \in d(E)$, $\forall n \in N$,

$\therefore d(E) = \mathbb{N}$.